

# $\mathcal{PT}$ -Symmetric Pseudo-Hermitian Relativistic Quantum Mechanics With a Maximal Mass

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## Abstract

The quantum-field model described by non-Hermitian, but a  $\mathcal{PT}$ -symmetric Hamiltonian is considered. It is shown by the algebraic way that the limiting of the physical mass value  $m \leq m_{max} = m_1^2/2m_2$  takes place for the case of a fermion field with a  $\gamma_5$ -dependent mass term ( $m \rightarrow m_1 + \gamma_5 m_2$ ). In the regions of unbroken  $\mathcal{PT}$  symmetry the Hamiltonian  $H$  has another symmetry represented by a linear operator  $\mathcal{C}$ . We exactly construct this operator by using a non-perturbative method. In terms of  $\mathcal{C}$  operator we calculate a time-independent inner product with a positive-defined norm. As a consequence of finiteness mass spectrum we have the  $\mathcal{PT}$ -symmetric Hamiltonian in the areas ( $m \leq m_{max}$ ), but beyond this limits  $\mathcal{PT}$ -symmetry is broken. Thus, we obtain that the basic results of the fermion field model with a  $\gamma_5$ -dependent mass term is equivalent to the Model with a Maximal Mass which for decades has been developed by V.Kadyshevsky and his colleagues. In their numerous papers the condition of finiteness of elementary particle mass spectrum was introduced in a purely geometric way, just as the velocity of light is a maximal velocity in the special relativity. The adequate geometrical realization of the limiting mass hypothesis is added up to the choice of (anti) de Sitter momentum space of the constant curvature.

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## 1 Introduction

In 1965 M. A. Markov [1] pioneered the hypotheses according to which the mass spectrum of the elementary particles should be cut off at the Planck mass  $m_{Planck} = 10^{19} GeV$  :

$$m \leq m_{Planck}. \quad (1)$$

The particles with the limiting mass  $m = m_{Planck}$ , named by the author "maximons" should play special role in the world of elementary particles. However, Markov's original condition (1) was purely phenomenological and he used standard field theoretical techniques even for describing the maximon.

Till recently one can see no reason why Standard Model (SM) should not be adequate up to value of order the Planck mass. But we are living in times, where many of the basic principles of physics are being challenged by need to go beyond SM. By now it is confirmed that dark matter exists and it consists of a large fraction of the energy density of the Universe.

In this connection a more radical approach was developed [2] - [14]. The Markov's idea about existence of a maximal value for the masses of the elementary particles has been understood as a new fundamental principle of Nature, which similarly to the relativistic and quantum postulates should be put in the grounds of quantum field theory (QFT). Doing this the condition of finiteness of the mass spectrum should be introduced by the relation:

$$m \leq M, \quad (2)$$

where the maximal mass parameter  $M$  called the "fundamental mass" is a new universal physical constant. Now objects for which  $m > M$  cannot be considered as elementary particles, as to them does not correspond a local field.

A new concept of a local quantum field has been developed on the ground of (2) and on simple geometric arguments, the corresponding Lagrangians were constructed and an adequate formulation of the principle of local gauge invariance has been found. It has been also demonstrated that the fundamental mass  $M$  in the new approach plays the role of an independent universal scale in the region of ultra high energies  $E \geq M$ .

The above-presented approach allows a simple geometric realization if one considers that the fundamental mass  $M$  is the curvature radius of the momentum anti de Sitter 4-space ( $\hbar = c = 1$ )

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 + p_5^2 = M^2. \quad (3)$$

For a free particle, for which  $p_0^2 - \vec{p}^2 = m^2$ , the condition (2) is automatically satisfied on the surface (3). In the approximation

$$|p_0|, |\vec{p}| \ll M, p_5 \cong M. \quad (4)$$

the anti de Sitter geometry does not differ from the Minkowski geometry in four dimensional pseudo-Euclidean  $p$ -space ("flat limit").

However, it is much less obvious that in the momentum 4-space (3) one may fully develop the apparatus of QFT, which after transition to configuration representation (with the help of a specific 5-dimensional Fourier transform) looks like a local field theoretical formalism in the four dimensional  $x$ -space [2] - [4]. It is fundamentally important that the new theory may be formulated in a gauge invariant way [5] - [8]. In other words, in the considered geometric approach there are conditions to construct an adequate generalization of the SM, which was called the Maximal Mass Model [9],[10].

Non-Hermitian quantum mechanics has recently created a lot of interest. This is due to the observation by Bender and Boettcher [19, 20] that with properly defined boundary conditions, the spectrum of the system described by the Hamiltonian  $H = p^2 + x^2(ix)^\nu$ ,  $\nu \geq 0$  is real positive and discrete. The reality of the spectrum is a consequence of unbroken  $\mathcal{PT}$  i.e. combined Parity  $\mathcal{P}$  and Time reversal  $\mathcal{T}$  invariance of the Hamiltonian,  $[H, PT]\psi = 0$  and the spectrum becomes partially complex when the PT symmetry is broken spontaneously [21, 22].

This new result has given rise to growing interest in the literature, see for examples, [21]-[32]. Past few years many non-Hermitian but  $\mathcal{PT}$  symmetric systems have been investigated including field theoretic models [28]-[32].

In an alternative approach [33]-[36], it has been shown that the reality of the spectrum of a non-Hermitian system is due to so called pseudo-Hermiticity properties of the Hamiltonian. A Hamiltonian is called  $\eta_0$  pseudo-Hermitian if it satisfies the relation

$$\eta_0 H \eta_0^{-1} = H^\dagger, \quad (5)$$

where  $\eta_0$  is some linear Hermitian and invertible operator. All PT symmetric non-Hermitian systems are pseudo-Hermitian where parity operator plays the role of  $\eta_0$ .

However, most of the previous works in the pseudo-Hermitian quantum mechanics have been carried out in the non-relativistic framework. One of the purpose of this paper is to extend the results of pseudo-Hermitian quantum mechanics for relativistic systems. Here we consider an example of pseudo-Hermitian Hamiltonian and show that the mass spectrum obtained by solving corresponding Dirac equation with a  $\gamma_5$ -mass term is not only real, but should be cut off  $m \leq m_{max}$ .

Particularly, the eigenstates of  $\mathcal{PT}$  symmetric non-Hermitian Hamiltonians, with real eigenvalues only, do not satisfy standard completeness relations. More importantly the eigenstates have negative norms if one takes the natural inner product associated with such systems defined as

$$\langle f | g \rangle_{\mathcal{PT}} = \int d^3x [\mathcal{PT}f(x)]g(x). \quad (6)$$

These problems are overcome by introducing a new symmetry  $\mathcal{C}$ , analogous to charge conjugation symmetry, associated with all such systems with equal number of negative and positive norm states [31, 32]. This allows to introduce an inner product structure associated with  $\mathcal{CPT}$  conjugate as

$$\langle f | g \rangle_{\mathcal{CPT}} = \int d^3x [\mathcal{CPT}f(x)]g(x), \quad (7)$$

for which the norms of the quantum states are positive definite and one gets usual completeness relation. As a result, the Hamiltonian and its eigenstates can be extended to complex domain so that the associated eigenvalues are real and underlying dynamics is unitary. Thus we have a fully consistent quantum theory for non-Hermitian but  $\mathcal{PT}$  invariant systems.

The norms of the state vectors, defined according to the modified rule of scalar product will be positive definite if we construct the  $\mathcal{C}$  operator. In Refs [30] it is shown that  $\mathcal{C}$  operator has the general form

$$\mathcal{C} = e^Q P, \quad (8)$$

where  $Q$  is a Hermitian operator and  $P$  is parity operator. However, unlike to [30] where operator  $Q$  has been obtained perturbatively we can construct  $\mathcal{C}$  operator immediately in closed form. Similarly the positive definite  $\eta$ -operator in the same way we define.

This paper is arranged as follows. The basic principles of the quantum field theory with the Maximal Mass are considered in section II. In section III, we formulate a new algebraic condition of an unbroken  $\mathcal{PT}$  symmetry. Then the particle mass finiteness in the theory with a  $\gamma_5$  mass term is obtained and the  $\mathcal{C}$  operator is exactly calculated by using the non-perturbative method. The positive definite  $\eta$  and  $\mathcal{C}$  -operators we also construct here. Last IV section reveals for conclusion and summary

## 2 The Quantum Field Model With a Maximal Mass

As for the mass of the particle  $m$ , this quantity is the Casimir operator of the *noncompact* Poincaré group and in the unitary representations of this group, used in QFT, they may have arbitrary values in the interval  $0 \leq m < \infty$ . In the SM one observe a great variety in the mass values. For example, t-quark is more than 300000 times heavier than the electron. In this situation the question naturally arises: up to what values of mass one may apply the concept of a local quantum field? Formally the contemporary QFT remains logically perfect scheme and its mathematical structure does not change at all up to arbitrary large values of quanta's masses. For instance, the free Klein-Gordon equation for the one component real scalar field  $\varphi(x)$  has always the form:

$$(\square + m^2)\varphi(x) = 0. \quad (9)$$

From here after standard Fourier transform:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int e^{-ip_\mu x^\mu} \varphi(p) d^4p \quad (p_\mu x^\mu = p^0 x^0 - \mathbf{p} \cdot \mathbf{x}), \quad (10)$$

we find the equation of motion in Minkowski momentum 4-space:

$$(m^2 - p^2)\varphi(p) = 0, \quad p^2 = p_0^2 - \mathbf{p}^2. \quad (11)$$

From geometrical point of view  $m$  is the radius of the "mass shell" hyperboloid:

$$m^2 = p_0^2 - \mathbf{p}^2, \quad (12)$$

where the field  $\varphi(p)$  is defined and in the Minkowski momentum space one may embed hyperboloids of the type (12) of arbitrary radius.

It is worth emphasizing that here, due to eq(2), the Compton wave length of a particle  $\lambda_C = \hbar/mc$  can not be smaller than the "fundamental length"  $l = \hbar/Mc$ . According to Newton and Wigner [18] the parameter  $\lambda_C$  characterizes the dimensions of the region of space in which a relativistic particle of mass  $m$  can be localized. Therefore the fundamental length  $l$  introduces into the theory an universal bound on the accuracy of the localization in space of elementary particles.

Let us go back to the free one component real scalar field we considered above (33 - 11). We shall suppose that its mass  $m$  satisfies the condition (2).

How should one modify the equations of motion in order that the existence of the bound (2) should become as evident as it is the limitation  $v \leq c$  in the special theory of relativity? In the latter case everything is explained in a simple way: the relativization of the 3-dimensional velocity space is equivalent to transition in this space from Euclidean to Lobachevsky geometry, realized on the 4-dimensional hyperboloid. Let us act in a similar way and change the 4-dimensional Minkowski momentum space, which is used in the standard QFT, to anti de Sitter momentum space, realized on the 5-hyperboloid:

$$p_0^2 - \mathbf{p}^2 + p_5^2 = M^2. \quad (13)$$

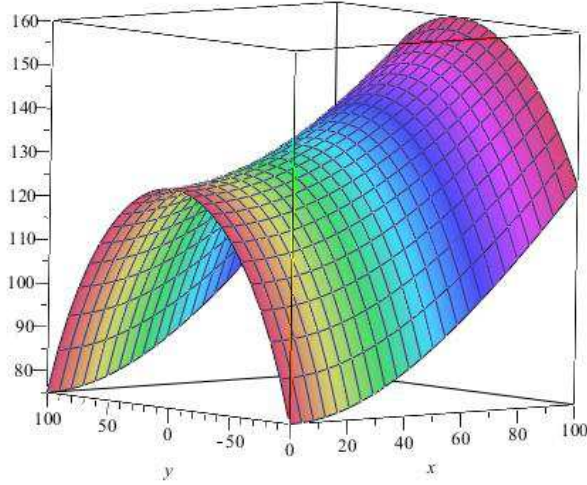


Figure 1: Curvature momentum space, realized on the hyperboloid  $p_0^2 - p_1^2 + p_5^2 = M^2$ , for  $M = 125 GeV$ .

In Fig. 1 we have the 3D-plot of  $P_0$  as function of  $P_1$  and  $P_5$ , for the case  $P_2 = P_3 = 0$  and maximal mass  $M = 125 GeV$ .

We shall suppose that in p-representation our scalar field is defined just on the surface (13), i.e. it is a function of five variables  $(p_0, \mathbf{p}, p_5)$ , which are connected by the relation (13):

$$\delta(p_0^2 - \mathbf{p}^2 + p_5^2 - M^2) \varphi(p_0, \mathbf{p}, p_5). \quad (14)$$

The energy  $p_0$  and the 3-momentum  $\mathbf{p}$  here preserve their usual sense and the mass shell relation (12) is satisfied as well. Therefore, for the considered field  $\varphi(p_0, \mathbf{p}, p_5)$  the condition (2) is always fulfilled.

Clearly in eq. (14) the specification of a single function  $\varphi(p_0, \mathbf{p}, p_5)$  of five variables  $(p_\mu, p_5)$  is equivalent to the definition of two independent functions  $\varphi_1(p)$  and  $\varphi_2(p)$  of the 4-momentum  $p_\mu$ :

$$\varphi(p_0, \mathbf{p}, p_5) \equiv \varphi(p, p_5) = \begin{pmatrix} \varphi(p, p_5) \\ \varphi(p, -p_5) \end{pmatrix} = \begin{pmatrix} \varphi_1(p) \\ \varphi_2(p) \end{pmatrix}, |p_5| = \sqrt{M^2 - p^2}. \quad (15)$$

The appearance of the new discrete degree of freedom  $p_5/|p_5|$  and the associated doubling of the number of field variables is a most important feature of the new approach. It must be taken into account in the search of the equation of motion for the free field in de Sitter momentum space. Because of the mass shell relation (12) the Klein - Gordon equation (11) should be also satisfied by the field  $\varphi(p_0, \mathbf{p}, p_5)$  :

$$(m^2 - p_0^2 + \mathbf{p}^2)\varphi(p_0, \mathbf{p}, p_5) = 0. \quad (16)$$

From our point of view this relation is unsatisfactory for 2 reasons:

1. It does not reflect the bounded mass condition (2).
2. It can not be used to determine the dependence of the field on the new quantum number  $p_5/|p_5|$  in order to distinguish between the components  $\varphi_1(p)$  and  $\varphi_2(p)$ .

Here we notice that, because of (13) eq.(16) may be written as:

$$(p_5 + M \cos \mu)(p_5 - M \cos \mu)\varphi(p, p_5) = 0, \quad \cos \mu = \sqrt{1 - \frac{m^2}{M^2}}. \quad (17)$$

Now, following the Dirac trick we postulate the equation of motion under question in the form:

$$2M(p_5 - M \cos \mu)\varphi(p, p_5) = 0. \quad (18)$$

Clearly, eq. (18) has none of the enumerated defects present in the standard Klein-Gordon equation (11). However, equation (11) is still satisfied by the field  $\varphi(p, p_5)$ .

From eqs. (18) and (15) it follows that:

$$\begin{aligned} 2M(|p_5| - M \cos \mu)\varphi_1(p) &= 0, \\ 2M(|p_5| + M \cos \mu)\varphi_2(p) &= 0, \end{aligned} \quad (19)$$

and we obtain:

$$\begin{aligned}\varphi_1(p) &= \delta(p^2 - m^2)\tilde{\varphi}_1(p) \\ \varphi_2(p) &= 0.\end{aligned}\tag{20}$$

Therefore, the free field  $\varphi(p, p_5)$  defined in anti de Sitter momentum space (13) describes the same free scalar particles of mass  $m$  as the field  $\varphi(p)$  in Minkowski p-space, with the only difference that now we necessarily have  $m \leq M$ . The two component structure (15) of the new field does not manifest itself on the mass shell, owing to (20). However, it will play an important role when the fields interact - i.e off the mass shell.

In the ordinary formalism the free Dirac operator

$$D(p) = p_\nu \gamma^\nu - m; \nu = 0, 1, 2, 3 \tag{21}$$

appears as a result of factorization of the Klein-Gordon wave operator

$$p_\nu^2 - m^2 = (p_\nu \gamma^\nu + m)(p_\nu \gamma^\nu - m). \tag{22}$$

Now instead of (21),(22) we obtain the following factorization formulas:

$$\begin{aligned}2M(p_5 + M \cos \mu) &= [\gamma^0 p_0 - \gamma \mathbf{p} - \\ \gamma^5(p_5 + M) - 2M \sin \mu/2] &[\gamma^0 p_0 - \gamma \mathbf{p} - \gamma^5(p_5 + M) + 2M \sin \mu/2].\end{aligned}\tag{23}$$

$$\begin{aligned}2M(p_5 - M \cos \mu) &= [\gamma^0 p_0 - \gamma \mathbf{p} \\ -\gamma^5(p_5 - M) + 2M \sin \mu/2] &[-\gamma^0 p_0 + \gamma \mathbf{p} + \gamma^5(p_5 - M) + 2M \sin \mu/2].\end{aligned}\tag{24}$$

There is instead of (21) we have the new expression for the Dirac operator

$$D(p, M) = p_\nu \gamma^\nu + (p_5 - M)\gamma^5 + 2M \sin(\mu/2). \tag{25}$$

It is easy to check that in the "flat approximation"

$$|p_\nu| \ll M, \quad m \ll M, p_5 \ll M, p_5 \simeq M.$$

both expressions (21),(25) coincide. But the amusing point is that the new Klein-Gordon operator  $2M(p_5 - \cos \mu)$  has one more decomposition into matrix factors:

$$2M(p_5 - \cos \mu) = [\gamma^0 p_0 - \gamma \mathbf{p} \tag{26}$$



$$-\gamma^5(p_5 + M) + 2M\cos\mu/2][-\gamma^0 p_0 + -\gamma\mathbf{p} + \gamma^5(p_5 + M) - 2M\cos\mu/2].$$

Therefore, if our approach is considered to be realistic, it may be assumed that in Nature there exists some exotic fermion field associated with the wave operator

$$D_{exotic}(p, M) = p_\nu\gamma^\nu + (p_5 + M)\gamma^5 - 2M\cos(\mu/2). \quad (27)$$

In contrast to (25) the operator  $D_{exotic}(p, M)$  dose not have a flat limit ( $M \rightarrow \infty$ ).

### 3 The Particle Mass Finiteness in The Theory With a $\gamma_5$ - Mass Term

Now we can consider (23),(24)in configuration space on the mass surface  $p_5 = \pm\sqrt{M^2 - m^2}$  ([9]).

For the case  $p_5 = -\sqrt{M^2 - m^2}$  we have

$$(p_0 - \alpha\mathbf{p} - \beta m_1 - \beta\gamma^5 m_2)\Psi_1(x, t, x_5) = 0. \quad (28)$$

$$(p_0 - \alpha\mathbf{p} + \beta m_1 - \beta\gamma^5 m_2)\Psi_2(x, t, x_5) = 0.$$

Analogously, for the case  $p_5 = \sqrt{M^2 - m^2}$  we can write

$$(p_0 - \alpha\mathbf{p} - \beta m_1 + \beta\gamma^5 m_2)\Psi_3(x, t, x_5) = 0. \quad (29)$$

$$(p_0 - \alpha\mathbf{p} + \beta m_1 + \beta\gamma^5 m_2)\Psi_4(x, t, x_5) = 0.$$

In this equations Dirac matrices  $\beta = \gamma_0$ ,  $\gamma^i = \beta\alpha^i$ .

Equation (28),(29) differ from each other only in their signs before the terms with  $m_1$  and  $m_2$ . As for their physical this equation are equivalent to the ordinary Dirac equations differing in their signs before the mass term. It is very important to note, that on the mass surface there are not the operators, which act on the coordinate  $x_5$  and this parameter can be taken equal to zero ([9]),([10]).

Any of the equivalent Hamiltonian from equations(23),(24) takes the form

$$\hat{H} = \vec{\alpha} \vec{p} + \hat{\beta} (m_1 + m_2\gamma_5) \quad (30)$$

We consider now the case of two-dimensional space-time. In (1+1)-dimensional space-time we adopt the conventions used in Ref. [37]:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (31)$$

With these definitions we have  $\gamma_0^2 = 1$  and  $\gamma_1^2 = -1$ . We also define

$$\gamma_5 = -\gamma_0\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (32)$$

so that  $\gamma_5^2 = 1$ .

It is easy to see that the Hamiltonian  $H_0 = \vec{\alpha} \cdot \vec{p} + \hat{\beta}(m)$  is Hermitian:  $H_0 = H_0^\dagger$ . Also,  $H_0$  is separately invariant under parity reflection and under time reversal:

$$\mathcal{P}H_0\mathcal{P} = H_0 \quad \text{and} \quad \mathcal{T}H_0\mathcal{T} = H_0.$$

Now let us consider a *non*-Hermitian Hamiltonian (30). The Hamiltonian  $H$  is not Hermitian because the  $m_2$  term changes sign under Hermitian conjugation. Also,  $H$  is not invariant under  $\mathcal{P}$  or under  $\mathcal{T}$  separately because the  $m_2$  term changes sign under each of these reflections. This sign change occurs because  $\hat{\beta}$  and  $\gamma_5$  anticommute. However,  $H$  is *invariant* under combined  $\mathcal{P}$  and  $\mathcal{T}$  reflection. Thus,  $H$  is  $\mathcal{PT}$ -symmetric:  $H^{\mathcal{PT}} = \mathcal{P}\mathcal{T}H\mathcal{P}\mathcal{T} = H$ .

The equation (29) can be transformed to the equation of the second order and as a result we have the Klein-Gordon equation

$$(\partial^2 + m^2)\psi(x, t) = 0, \quad (33)$$

where

$$m^2 = m_1^2 - m_2^2. \quad (34)$$

It is easy to see the physical mass  $m$  that propagates under this equation is real when the inequality

$$m_1^2 \geq m_2^2 \quad (35)$$

is satisfied. This inequality (35) was considered by C.Bender et al. in Ref.[30] as the basic condition.

Now we prove that the parameters  $m_1$  and  $m_2$  have auxiliary nature because that assume an ambiguous definition. Taking into account (34), we can write the following obvious inequality

$$(m - m_2)^2 \geq 0, \quad (36)$$

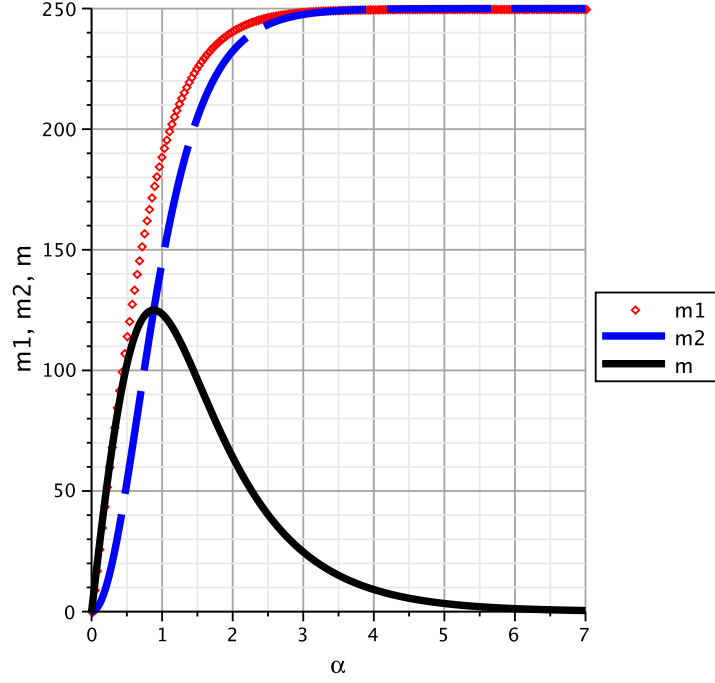


Figure 2: The values of parameters  $m$ ,  $m_1, m_2$  as functions of  $\alpha$ ;  $m(\max)=125$  GeV.

from which to obtain

$$m \leq \frac{m_1^2}{2m_2} = m_{max}. \quad (37)$$

The conditions (34),(35) and (37) are satisfied automatically if we introduce the following parametrization:

$$m_1 = m \cosh(\alpha); \quad m_2 = m \sinh(\alpha). \quad (38)$$

Indeed, from (37) and (38) we can also define

$$m = 2m_{max} \frac{\sinh \alpha}{\cosh^2 \alpha}, \quad (39)$$

$$m_1 = 2m_{max} \tanh \alpha, \quad (40)$$

$$m_2 = 2m_{max} \tanh^2 \alpha. \quad (41)$$

For this mass the all conditions (34), (35) and (37) are realized. Fig. (2) displays parameters  $m$ ,  $m_1$  and  $m_2$  as functions of  $\alpha$  are presented. Value entry of the maximal mass is  $125\text{GeV}$ . The values of parameters  $m$ ,  $m_1, m_2$  describes the propagation of particle having positive mass  $m$  can be varied in a wide range. The  $m$  reaches a high ( $m = m_{max}$ ) in the point  $\alpha_0 = 0.881$ .

The equation (37) can be represented in the form

$$\frac{m}{2m_{max}} = \tanh(\alpha) \sqrt{1 - \tanh^2(\alpha)}. \quad (42)$$

The solution of (42) is

$$\tanh(\alpha) = \sqrt{\frac{1 \pm \sqrt{1 - m^2/m_{max}^2}}{2}}. \quad (43)$$

Two signs of square root can be interpreted by the following way: we have the dual branches of the values  $m_1$  and  $m_2$  as functions of the physical mass  $m$ .

This is the reason to insert a new definitions for mass. Using (40),(41) we have

$$m_1 = \sqrt{2}m_{max} \sqrt{1 - \sqrt{1 - m^2/m_{max}^2}}; \quad (44)$$

$$m_2 = m_{max} \left(1 - \sqrt{1 - m^2/m_{max}^2}\right); \quad (45)$$

$$m_3 = \sqrt{2}m_{max} \sqrt{1 + \sqrt{1 - m^2/m_{max}^2}} \quad (46)$$

$$m_4 = m_{max} \left(1 + \sqrt{1 - m^2/m_{max}^2}\right). \quad (47)$$

Fig. (3) displays parameters  $m_1$ ,  $m_2, m_3$  and  $m_4$  as functions of  $m$  values are presented. The region of unbroken  $\mathcal{PT}$  symmetry  $m \leq m_{max}$ . For these values of the parameters  $m_1$  and  $m_2$ , the new Dirac equation describes the propagation of particles having the real mass. The special case of Hermiticity is obtained on the line  $m = m_{max}$ , which is achieved at the edge of the region of unbroken  $\mathcal{PT}$  symmetry. In this point we have  $m_1 = m_3 = \sqrt{2}m_{max}$  and  $m_2 = m_4 = m_{max}$ .

There is no difficulty in understanding that a new mass parameters are really satisfied to conditions (34), (35). In particular, we have  $m_3 \geq m_4$  and

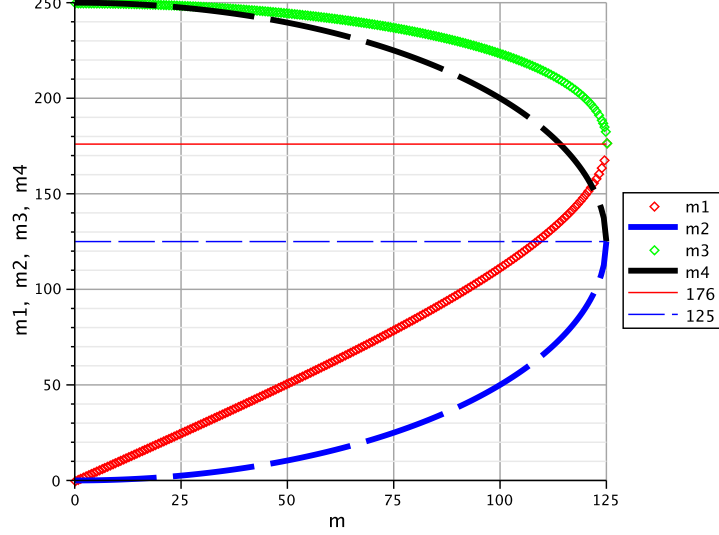


Figure 3: The values of parameters  $m_1, m_2, m_3, m_4$  as functions of  $m$ ;  $M = 125 \text{ GeV}$ .

$$m_3^2 - m_4^2 = m^2. \quad (48)$$

The condition (37) and an ambiguous definition of  $m_1, m_2$  are in agreement with the Kadyshevsky's basic principal of the geometrical scheme [9],[10].

This approach may be used for the calculation of the  $\mathcal{C}$  operator associated with the  $\mathcal{PT}$ -symmetry of Hamiltonian. We begin by letting (38) and rewriting the mass terms in Hamiltonian in the form

$$\hat{\beta}(m_1 + m_2 \gamma_5) = \hat{\beta}m(\cosh \alpha + \gamma_5 \sinh \alpha) = \hat{\beta}m \exp(\gamma_5 \alpha). \quad (49)$$

Then, Hamiltonian  $\hat{H}$  is given by

$$\hat{H} = \hat{\vec{\alpha}} \vec{p} + \hat{\beta}m \exp(\gamma_5 \alpha), \quad (50)$$

and Hermitian-conjugate Hamiltonian takes the form

$$\hat{H}^+ = \hat{\vec{\alpha}} \vec{p} + \hat{\beta}m \exp(-\gamma_5 \alpha). \quad (51)$$

Next, we can write

$$e^{\hat{\alpha}\gamma_5/2} \hat{H} = \left( \hat{\vec{\alpha}} \vec{p} + \hat{\beta}m \right) e^{\hat{\alpha}\gamma_5/2} = \hat{H}_0 e^{\hat{\alpha}\gamma_5/2}, \quad (52)$$

where the sign before the mass term  $m_2$  change occurs because the  $\gamma_5$  and  $\beta$  anticommute and

$$\hat{H}_0 = \vec{\hat{\alpha}} \vec{p} + \hat{\beta}m$$

is the ordinary Dirac Hamiltonian.

For Hamiltonian  $\hat{H}^+$  we also have

$$e^{-\hat{\alpha}\gamma_5/2}\hat{H}^+ = \left(\vec{\hat{\alpha}} \vec{p} + \hat{\beta}m\right)e^{-\hat{\alpha}\gamma_5/2} = \hat{H}_0e^{-\hat{\alpha}\gamma_5/2}. \quad (53)$$

It is easy to see from (52),(53) Hermitian Hamiltonian  $\hat{H}_0$  ( $\hat{H}_0 = \hat{H}_0^+$ ) and  $\hat{H}, \hat{H}^+$  are related by similarity transformations

$$\hat{H}_0 = e^{\frac{\gamma_5\alpha}{2}} \hat{H} e^{-\frac{\gamma_5\alpha}{2}}. \quad (54)$$

$$\hat{H}_0 = e^{-\frac{\gamma_5\alpha}{2}} \hat{H}^+ e^{\frac{\gamma_5\alpha}{2}}. \quad (55)$$

From (54),(55) we have

$$e^{-Q}\hat{H}e^Q = \hat{H}^+, \quad (56)$$

where

$$Q = -\alpha\gamma_5.$$

In Refs [28],[30] it is shown that the  $\mathcal{C}$  operator has the general form (8) and in the case of (1+1)-dimensional space-time,  $\mathcal{C}$  operator for the model with  $\gamma_5$ -mass term can be presented as

$$\mathcal{C} = \begin{pmatrix} 0 & \frac{m_1-m_2}{m} \\ \frac{m_1+m_2}{m} & 0 \end{pmatrix}. \quad (57)$$

One make sure that the operator  $\mathcal{C}$  is satisfied to the following system of three algebraic conditions:

$$\mathcal{C}^2 = 1, \quad (58)$$

$$[\mathcal{C}, \mathcal{PT}] = 0, \quad (59)$$

$$[\mathcal{C}, H] = 0. \quad (60)$$

By solving these three simultaneous equations for the operator  $\mathcal{C}$ , one obtains an inner product with respect to which  $H$  is self-adjoint.

In Ref.[34] it is shown that the square root of the positive operator

$$\eta \equiv \sqrt{e^{-Q}}$$

can be used to contract a Hermitian Hamiltonian  $\hat{H}_0$  that corresponds to the non-Hermitian Hamiltonian  $\hat{H}$ . From(54),(55) we can obtain

$$\eta = e^{\alpha\gamma_5/2}. \quad (61)$$

Now we construct the norm of any state for considered model using  $\mathcal{CPT}$ -symmetry. For arbitrary vector

$$\Psi = \begin{pmatrix} x + iy \\ u + iv \end{pmatrix},$$

we have

$$\mathcal{CPT}\Psi = \left( \frac{m_1 - m_2}{m}(x - iy), \frac{m_1 + m_2}{m}(u - iv) \right).$$

Then

$$\langle \mathcal{CPT}\Psi | \Psi \rangle = \frac{m_1 - m_2}{m}(x^2 + y^2) + \frac{m_1 + m_2}{m}(u^2 + v^2), \quad (62)$$

is explicitly non negative, because  $m_1 \geq m_2$ .

## 4 Conclusion

The investigations given in the previous sections show that the Dirac Hamiltonian of a particle with the a  $\gamma_5$ -dependent mass term ( $m \rightarrow m_1 + \gamma_5 m_2$ ) is non-Hermitian but a  $\mathcal{PT}$  symmetric. It is shown by the algebraic way that the limiting of the physical mass value  $m_{max} = m_1^2/2m_2$  takes place. In the regions of unbroken ( $m \leq m_{max}$ )  $\mathcal{PT}$  symmetry the Hamiltonian  $H$  has another symmetry represented by a linear operator  $\mathcal{C}$  (8).

We exactly construct this operator (57) by using a non-perturbative method. In terms of  $\mathcal{C}$  operator we calculate a time-independent inner product with a positive-defined norm. As a consequence of finiteness mass spectrum we have the  $\mathcal{PT}$ -symmetric Hamiltonian in the areas  $m \leq m_{max}$ , but beyond this limits  $\mathcal{PT}$ -symmetry is broken.

We proved that the parameters  $m_1$  and  $m_2$  have auxiliary nature because assume an ambiguous definition. This fact can be also confirmed by making

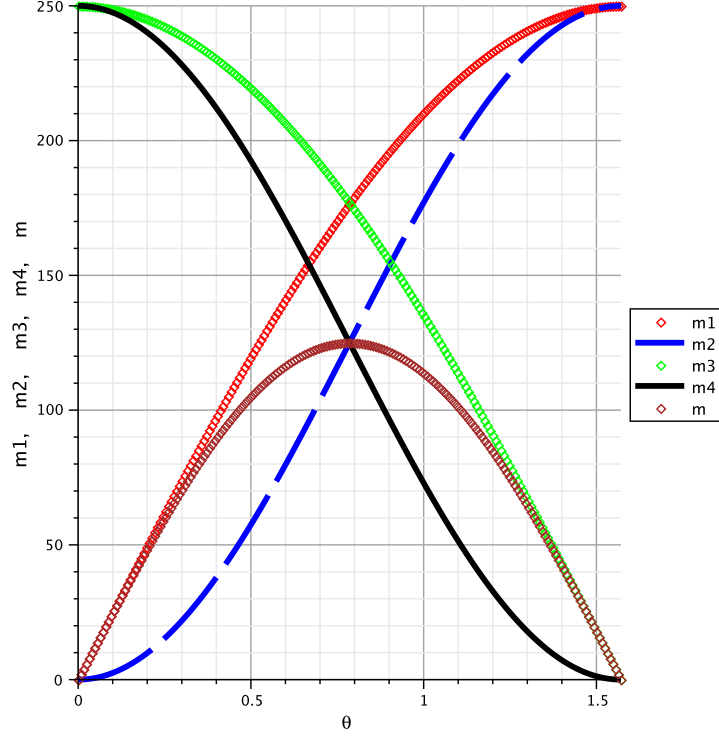


Figure 4: The values of parameters  $m_1, m_2, m_3, m_4, m$  as functions of  $\theta$ ;  $M = 125 \text{ GeV}$ .

a comparison between ordinary(having a flat limit) and "exotic fermion field" which dose not have a limit when  $m_{max} \rightarrow \infty$ . Let write a new definitions of mass  $m_3$  and  $m_4$  for the exotic field to satisfy conditions

$$m_3^2 = m^2 + m_4^2 \quad (63)$$

$m = m_3 \sin \theta$ ;  $m_4 = m_3 \cos \theta$ . Then we have

$$m_{max} = \frac{m}{\sin 2\theta}; \quad m_3 = m_{max} \cos \theta; \quad m_4 = m_{max} \cos^2 \theta; \quad 0 \leq \theta \leq \pi/2.$$

Analogously, for ordinary field ( $m_1^2 = m^2 + m_2^2$ ) we have  $m = m_1 \cos \theta$ ;  $m_2 = m_1 \sin \theta$ ; and can obtain

$$m_{max} = \frac{m}{\sin 2\theta}; \quad m_1 = m_{max} \sin \theta; \quad m_2 = m_{max} \sin^2 \theta; \quad 0 \leq \theta \leq \pi/2.$$



Fig. (4) displays parameters  $m_1$ ,  $m_2, m_3$  and  $m_4$  as functions of  $\theta$  values are presented. The region of unbroken  $\mathcal{PT}$  symmetry  $m \leq m_{max}$  and in term of the parameter  $\theta$  one can write as

$$0 \leq \theta \leq \pi/2.$$

For these values of the parameters  $m_1$  and  $m_2$  the new Dirac equation describes the propagation of particles having the real mass. The special case of Hermiticity is obtained on the line  $m = m_{max}$ , which is achieved at the center of the region of unbroken  $\mathcal{PT}$  symmetry  $\theta_0 = \pi/4$ . In this point we have  $m_1 = m_3 = \sqrt{2}m_{max}$  and  $m_2 = m_4 = m_{max}$ .

Thus, we obtain that the basic results of the fermion field model with a  $\gamma_5$ -dependent mass term is equivalent to the Model with a Maximal Mass which for decades has been developed by V.Kadyshevsky and his colleagues [2] - [14]. In particular, the exotic fermion field (27) associated with the new wave operator which does not have a limit when  $M \rightarrow \infty$  was investigated. The polarization properties of such the exotic fermion field differ sharply from the standard ones. It is tempting to think that quanta of the exotic fermion field have a relation to the structure of the "dark matter".

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